Counterexamples Concerning the Diagonal Elements of Normal Matrices

Robert C. Thompson
Mathematics Department and the Algebra Institute
University of California at Santa Barbara
Santa Barbara, California 93106

Submitted by Richard A. Brualdi

ABSTRACT

An infinite class of conditions known to be satisfied by the diagonal elements of a normal matrix with prescribed spectrum is shown to be independent of other known conditions satisfied by the diagonal elements but nevertheless insufficient to characterize the diagonal.

A long-open question in linear algebra is the relationship between the diagonal elements and the eigenvalues of a normal matrix. Let $d_1,\dots,d_n$ be the diagonal elements of a normal matrix N, $\lambda_1,\dots,\lambda_n$ its eigenvalues, and take

$$d=(d_1,\dots,d_n)^T,\qquad \lambda=(\lambda_1,\dots,\lambda_n)^T$$

to be the column vectors formed from the d_i and the λ_i. It is quite elementary that

$$d=S\lambda, \tag{1}$$

where $S=[|u_{ij}|^2]$ is a doubly stochastic matrix of unitary type, that is, its entries are the squares of the moduli of the corresponding entries of a unitary matrix $[u_{ij}]$. (In fact, the columns of $[u_{ij}]$ are the eigenvectors of N.) This is more restrictive than asserting that $d=S\lambda$ with S merely doubly stochastic, since (for $n\geqslant 3$) not every doubly stochastic matrix is of unitary type.

Given complex numbers $d_1,\dots,d_n$ and $\lambda_1,\dots,\lambda_n$, the question to be addressed in this paper (but *not* solved) is this: What conditions must the d_i and the λ_i satisfy in order that they be the diagonal elements and the eigenvalues of some normal matrix N? In view of the last paragraph,

52 Vanderbilt Ave., New York, NY 10017 0024-3795/83/010285-07$3.00

conditions should first be imposed to ensure that d is a doubly stochastic transform of λ, then further conditions added to ensure that the doubly stochastic transform will be of unitary type.

Now, conditions to force d to be a doubly stochastic transform of λ were given by S. Sherman many years ago [5], and recently reworked in [2]. Although somewhat awkward to use in individual cases, we shall nevertheless assume that Sherman's conditions have been verified, and now d is known to be a doubly stochastic transform of λ. What additional conditions have to be added to Sherman's to ensure that the transforming doubly stochastic matrix may be taken of unitary type?

In order to set the scene, we now digress to a brief discussion of singular values. If A is a matrix, unitary matrices U and V always exist such that UAV is diagonal with real, nonnegative, diagonal elements, called the singular values of A. An open question for some years was the nature of the relationship between the diagonal elements of a not necessarily diagonal matrix and its singular values. If the diagonal elements are numbered so that $|d_1| \geqslant \cdots \geqslant |d_n|$, and if $s_1 \geqslant \cdots \geqslant s_n$ are the singular values, the necessary and sufficient relations are a set of inequalities:

$$|d_1| + \cdots + |d_k| \leqslant |\lambda_1| + \cdots + |\lambda_k|, \qquad 1 \leqslant k \leqslant n, \tag{3}$$

and

$$|d_1| + \cdots + |d_{n-1}| - |d_n| \leqslant s_1 + \cdots + s_{n-1} - s_n. \tag{4}$$

This theorem was found by the present author [7], found again very soon thereafter by Sing [6], and partly found by Miranda [unpublished].

Now, the singular values of a normal matrix N are the moduli of its eigenvalues. And if N is normal, so is $zI - N$ for every complex number z, where I is the identity matrix. Applying the singular value–diagonal element inequality (4) to $zI - N$, we conclude that:

THEOREM 1. *If a normal matrix has diagonal elements $d_1, \ldots, d_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$, then for every complex number z,*

$$\begin{aligned} &|z - d_1| + \cdots + |z - d_n| - 2\min\{|z - d_1|, \ldots, |z - d_n|\} \\ &\quad \leqslant |z - \lambda_1| + \cdots + |z - \lambda_n| - 2\min\{|z - \lambda_1|, \ldots, |z - \lambda_n|\}. \end{aligned} \tag{5}$$

This theorem was already stated in [7], but no attempt was made there to address the two questions to which it immediately leads. These are: (a) is (5) a genuine additional condition, that is, not implied by $d = S\lambda$ with S doubly stochastic, and (b) if it is, is it the desired additional condition that characterizes the diagonal of a normal matrix with prescribed spectrum? The objectives of this paper may now be stated. They are (i) to show that (5) is indeed not implied by $d = S\lambda$ with S doubly stochastic, and (ii) that (5) is unfortunately not strong enough to characterize the diagonal of a normal matrix.

To achieve these objectives, we shall produce two 3×3 examples. In each, we have $d = S\lambda$ with S a uniquely determined doubly stochastic matrix, not of unitary type, so that the d_i cannot be the diagonal elements of a normal matrix with spectrum λ_i. In the first example, the condition (5) will fail for at least one z, demonstrating that (5) is not a consequence of $d = S\lambda$. In the second, (5) will hold for every z, without exception, and this will show that $d = S\lambda$ and (5) are inadequate to characterize the diagonal elements of a normal matrix with given spectrum.

EXAMPLE 1. Let $\lambda_1, \lambda_1, \lambda_3$ be the vertices of an equilateral triangle in the complex plane, and put

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then for $z = d_2 = (\lambda_1 + \lambda_3)/2$, the condition (5) fails.

EXAMPLE 2. Let ϵ be a small positive number, and take S_ϵ to be the following doubly stochastic matrix:

$$S_\epsilon = \begin{bmatrix} \frac{1}{4}+\epsilon & \frac{3}{8}+\frac{1}{4}\sqrt{2}-\epsilon & \frac{3}{8}-\frac{1}{4}\sqrt{2} \\ \frac{1}{4} & \frac{3}{8}-\frac{1}{4}\sqrt{2} & \frac{3}{8}+\frac{1}{4}\sqrt{2} \\ \frac{1}{2}-\epsilon & \frac{1}{4}+\epsilon & \frac{1}{4} \end{bmatrix}.$$

For $\epsilon = 0$ this matrix is of unitary type, and in fact we then have $s_{ij} = u_{ij}^2$, where U is this real, orthogonal matrix:

$$U = \begin{bmatrix} \frac{1}{2} & \left(\frac{3}{8}+\frac{1}{4}\sqrt{2}\right)^{1/2} & \left(\frac{3}{8}-\frac{1}{4}\sqrt{2}\right)^{1/2} \\ \frac{1}{2} & -\left(\frac{3}{8}-\frac{1}{4}\sqrt{2}\right)^{1/2} & -\left(\frac{3}{8}+\frac{1}{4}\sqrt{2}\right)^{1/2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

However, for $\epsilon > 0$, S_ϵ is not of unitary type, since the following necessary condition for this property is not satisfied:

$$(s_{11}s_{21})^{1/2} \leqslant (s_{12}s_{22})^{1/2} + (s_{13}s_{23})^{1/2}.$$

(This condition arises from the orthogonality of the first two rows of any unitary matrix underlying S_ϵ.)

Take $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = i = (-1)^{1/2}$. Then, as functions of ϵ, the entries of $d = S_\epsilon \lambda$ are

$$\begin{aligned} d_1 &= d_1(\epsilon) = \tfrac{1}{8} + \tfrac{1}{4}\sqrt{2} - 2\epsilon + i(\tfrac{3}{8} - \tfrac{1}{4}\sqrt{2}), \\ d_2 &= d_2(\epsilon) = \tfrac{1}{8} - \tfrac{1}{4}\sqrt{2} \qquad\quad + i(\tfrac{3}{8} + \tfrac{1}{4}\sqrt{2}), \\ d_3 &= d_3(\epsilon) = -\tfrac{1}{4} + 2\epsilon \qquad\quad + i\tfrac{1}{4}. \end{aligned}$$

Let

$$\begin{aligned} f_\epsilon(z) = {} & |z-\lambda_1| + |z-\lambda_2| + |z-\lambda_3| - 2\min\{|z-\lambda_1|, |z-\lambda_2|, |z-\lambda_3|\} \\ & - \{|z-d_1| + |z-d_2| + |z-d_3| - 2\min\{|z-d_1|, |z-d_2|, |z-d_3|\}\}. \end{aligned}$$

We are going to prove that if ϵ is positive and sufficiently small, then $f_\epsilon(z) > 0$ for every complex number z, and yet d_1, d_2, d_3 are not diagonal elements of any normal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. This will show that $d = S\lambda$ and (5) do not imply that d is the diagonal of a normal matrix with spectrum λ. Here are the steps in the proof.

We first show that if $K = 10^8$, $\epsilon_0 = 0.001$, and $c = 10^{-4}$, then for every ϵ with $0 \leqslant \epsilon \leqslant \epsilon_0$, and every z with $|z| \geqslant K$, we have $f_\epsilon(z) \geqslant c$. To do this, we begin by observing that if a is a complex number with $|a| \leqslant 1$, and if z is in polar form, $z = re^{i\theta}$ [$r > 1$, θ real], then

$$|z-a| = r - \cos\theta \operatorname{Re} a - \sin\theta \operatorname{Im} a + \frac{1}{2r}E \tag{6}$$

with an error E satisfying

$$|E| \leqslant \frac{1}{(1-1/r)^3}.$$

This is just an expansion of $|z-a|$ in powers of $1/r$. (The calculation starts by using the law of cosines to express $|z-a|$ in terms of $|z|$ and $|a|$.)

Now, if the complex plane is divided into regions according to the λ_i and d_j nearest z (for fixed ϵ), it will be found that there are nine regions (eight when $\epsilon = 0$), of which six are unbounded and one (for small nonzero ϵ) is bounded but extends far from the origin. In each of these six or seven regions, $f_\epsilon(z)$ takes a different form. Applying the expansion (6) to each term in $f_\epsilon(z)$, it is possible in a routine way to show that $f_\epsilon(z) \geqslant c$ if $|z| \geqslant K$, for all $\epsilon \leqslant \epsilon_0$. This proves our first assertion.

Next, let

$$A = UDU^*, \tag{7}$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. This matrix A is normal, and also complex symmetric. Its diagonal elements are $d_1(0), d_2(0), d_3(0)$. One may directly verify that if $z = d_1(0)$, or if $z = d_2(0)$, or if $z = d_3(0)$, then $f_0(z) > 0$. We now show that $f_0(z) > 0$ for all other z.

To do this, we fix z and note that $B = zI - A$ has nonzero diagonal elements. Let

$$\Delta = \operatorname{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma})$$

be diagonal and unitary, with the angles α, β, γ so chosen that

$$C = \Delta B$$

has its two diagonal elements of largest moduli real and positive, and its diagonal element of smallest modulus real and negative. Let $\gamma_1, \gamma_2, \gamma_3$ be these diagonal elements of C, and let $s_1 \geqslant s_2 \geqslant s_3$ be the singular values of C. Then $f_0(z) \geqslant 0$ is the same as

$$|\gamma_1| + |\gamma_2| - |\gamma_3| \leqslant s_1 + s_2 - s_3, \tag{8}$$

and by the first theorem of [7] this is true. So $f_0(z) \geqslant 0$. Moreover, if $f_0(z) = 0$, we would have equality in (8). But, by the proofs of Lemmas 3, 4, and 5 of [7], this would mean that C is Hermitian. Stating this fact in terms of $zI - A$, we see that $f_0(z) = 0$ with z not a diagonal element of A implies that

$$e^{i\alpha} a_{12} = \overline{e^{i\beta} a_{21}}, \qquad e^{i\alpha} a_{13} = \overline{e^{i\gamma} a_{31}}, \qquad e^{i\beta} a_{23} = \overline{e^{i\gamma} a_{32}}, \tag{9}$$

for certain angles α, β, γ.

We have explicit values for $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$, found from (7). The equations (9) yield three congruences (mod 2π) for α, β, γ, leading to

$$\alpha \equiv -\arg a_{12} - \arg a_{13} + \arg a_{23} \pmod{\pi},$$

$$\beta \equiv -\arg a_{12} + \arg a_{13} - \arg a_{23} \pmod{\pi},$$

$$\gamma \equiv \arg a_{12} - \arg a_{13} - \arg a_{23} \pmod{\pi}.$$

Here, $\alpha \equiv -\arg(z - d_1) \bmod \pi$, $\beta \equiv -\arg(z - d_2) \bmod \pi$, $\gamma \equiv -\arg(z - d_3)$ mod π, and so these conditions force z to lie at the intersection of three straight lines, through d_1 with inclination $-\alpha$, through d_2 with inclination $-\beta$, and through d_3 with inclination $-\gamma$. But it will be found that these three lines do not have a common intersection point. A contradiction has resulted from the assumption $f_0(z) = 0$.

Thus, for all z, we have $f_0(z) > 0$. Let c_0 be the minimum of $f_0(z)$ on the compact set $|z| \leqslant K$; then $f_0(z) \geqslant c_0 > 0$. Furthermore,

$$\min_{\substack{z \\ |z| \leqslant K}} f_\epsilon(z)$$

is a continuous function of ϵ, and since $c_0 > 0$, there exists a value $\epsilon_0' > 0$ such that

$$\min_{0 \leqslant \epsilon \leqslant \epsilon_0'} \min_{|z| \leqslant K} f_\epsilon(z) \geqslant \tfrac{1}{2} c_0 > 0. \tag{10}$$

Now, take ϵ to be any value with $0 < \epsilon \leqslant \min(\epsilon_0, \epsilon_0')$. For this ϵ, and for every z, we infer from (10) and from $f_\epsilon(z) \geqslant c$ for $|z| \geqslant K$, that

$$f_\epsilon(z) \geqslant \min\left(\tfrac{1}{2} c_0, c\right) > 0.$$

For this value of ϵ we thus have $f_\epsilon(z) \geqslant 0$ for every z, and yet the entries of $d = S_\epsilon \lambda$ are not the diagonal entries of any normal matrix with spectrum λ, since S_ϵ is not of unitary type and is the only doubly stochastic matrix S for which $d = S\lambda$.

THEOREM 2. *While (5), for all complex numbers z, is a necessary condition to be satisfied by the diagonal elements and eigenvalues of a normal matrix, and is a condition not implied by $d = S\lambda$, S doubly stochastic, it is not a sufficiently strong supplement to $d = S\lambda$ to characterize the diagonal elements of a normal matrix with prescribed spectrum.*

The geometry underlying (5) seems very well hidden. Even in the 3×3 case, where a geometrical discussion of the diagonal of a normal matrix was given by Williams [9], the geometry of (5) is far from evident. Other papers, somewhat similar in spirit to our discussion above, are by Lerer [3] and by Au-Yeung and Poon [1]. See also Poon [4].

The inequalities (3), which seem to have been ignored, are in fact a consequence of $d = S\lambda$, and indeed one may also deduce $|z-d_1|+\cdots+|z-d_k|\leqslant|z-\lambda_1|+\cdots+|z-\lambda_k|$ for all k and all complex z, where the numbering of the d_i and the λ_i now depends on z.

REMARK. The above Theorem 2 constitutes a solution of the research problem posed in [8].

Preparation of this paper was partly supported by an AFOSR grant.

REFERENCES

1 Y. H. Au-Yeung and Y. T. Poon, 3×3 orthostochastic matrices and the convexity of the generalized numerical range, *Linear Algebra Appl.* 27:69–79 (1979).
2 H. Komiya, Doubly stochastic matrices and complex vectors, *Linear Algebra Appl.*, to appear.
3 L. E. Lerer, On the diagonal elements of normal matrices, *Mat. Issled.* 2:156–163 (1967).
4 Y. T. Poon, The generalized k numerical range, *Linear and Multilinear Algebra* 9:181–186 (1980).
5 S. Sherman, Doubly stochastic matrices and complex vector spaces, *Amer. J. Math.* 77:245–246 (1955).
6 F. Y. Sing, Some results on matrices with prescribed diagonal elements and singular values, *Canad. Math. Bull.* 19:89–92 (1976).
7 R. C. Thompson, Singular values, diagonal elements, and convexity, *SIAM J. Appl. Math.* 32:39–63 (1977).
8 R. C. Thompson, Research problem, *Linear and Multilinear Algebra* 10:263 (1981).
9 J. P. Williams, On compressions of matrices, *J. London Math. Soc.* 3:526–532 (1971).

Received 9 April 1982; revised 10 May 1982